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A New Proof of Teljakowskii's Theorem

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1. INTRODUCTION

In 1966 Teljakowskii [4] proved the following strengthening of Timan's theorem:

THEOREM 1.1. *For every function f continuous on $[-1, 1]$ and any natural n there is an algebraic polynomial $p_n(f; \cdot)$ of degree n such that for all $t \in [-1, 1]$,*

$$|f(t) - p_n(f; t)| \leqslant M\omega(f; \sqrt{1-t^2}/n), \tag{1.1}$$

where M is an absolute constant and $\omega(f; \delta)$ is the modulus of continuity of f.

A first indication about the order of the constant M was given by Saxena [3] in the following theorem:

THEOREM 1.2. *For every function f continuous on $[-1, 1]$ and any natural n there is an algebraic polynomial $A_n(f; \cdot)$ of degree $4n+2$ such that for all $t \in [-1, 1]$,*

$$|f(t) - A_n(f; t)| \leqslant 1285\omega(f; \sqrt{1-t^2}/n). \tag{1.2}$$

In the present paper we shall prove

THEOREM 1.3. *For every function f continuous on $[-1, 1]$ and any natural n, $n \geqslant 10$, there is an algebraic polynomial $\bar{H}_n(f; \cdot)$ of degree $3n-3$ such that for all $t \in [-1, 1]$,*

$$|f(t) - \bar{H}_n(f; t)| \leqslant 10\omega(f; \sqrt{1-t^2}/n). \tag{1.3}$$

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2. Proof of Theorem 1.3

We shall first describe the construction of the polynomials $\bar{H}_n(f; \cdot)$. Suppose a polynomial $H_n(f; \cdot)$ of degree $3n-3$ is given by a convolution-type operator of form

$$H_n(f; t) := \frac{1}{\pi} \int_{\pi}^{\pi} f(\cos(\arccos t + v))\, K_{3n-3}(v)\, dv \tag{2.1}$$

with the kernel (cf. [2, p. 14])

$$K_{3n-3}(v) = \frac{10}{n(11n^4 + 5n^2 + 4)} \left(\frac{\sin(nv/2)}{\sin(v/2)}\right)^6. \tag{2.2}$$

Now set

$$\bar{H}_n(f; t) := H_n(f; t) + \left\{\frac{t+1}{2}[f(1) - H_n(f; 1)] + \frac{1-t}{2}[f(-1) - H_n(f; -1)]\right\}. \tag{2.3}$$

It was proved in [1] that for every $f \in C[-1, 1]$ and any $n \in \mathbb{N}$ the inequality

$$|H_n(f; t) - f(t)| \leqslant 2\omega\left(f; \sqrt{\frac{30}{11}}\frac{|t|}{n^2} + \sqrt{\frac{20}{11}}\frac{\sqrt{1-t^2}}{n}\right) \tag{2.4}$$

holds for all $t \in [-1, 1]$.

Case (a) $1/n \leqslant \sqrt{1-t^2}$. On account of inequality (2.4) we have

$$\begin{aligned} |f(t) - \bar{H}_n(f; t)| &\leqslant |f(t) - H_n(f; t)| + \frac{(t+1)}{2}|f(1) - H_n(f; 1)| \\ &\quad + \frac{(1-t)}{2}|f(-1) - H_n(f; -1)| \\ &\leqslant 2\omega\left(f; \sqrt{\frac{30}{11}}\frac{|t|}{n^2} + \sqrt{\frac{20}{11}}\frac{\sqrt{1-t^2}}{n}\right) + 2\omega\left(f; \sqrt{\frac{30}{11}}\frac{1}{n^2}\right) \\ &\leqslant 10\omega(f; \sqrt{1-t^2}/n). \end{aligned}$$

Case (b) $\sqrt{1-t^2} \leqslant 1/n$, $t \geqslant 0$. In this case we obtain from (2.3)

$$\begin{aligned} f(t) - \bar{H}_n(f; t) &= [f(t) - f(1)] - [H_n(f; t) - H_n(f; 1)] \\ &\quad + \frac{1-t}{2}\{[f(1) - H_n(f; 1)] - [f(-1) - H_n(f; -1)]\} \end{aligned}$$

and thus by means of (2.4)

$$
\begin{aligned}
&|f(t)-\bar{H}_n(f;t)| \\
&\quad \leqslant |f(t)-f(1)| + |H_n(f;t)-H_n(f;1)| \\
&\qquad + \frac{(1-t)}{2}\{|f(1)-H_n(f;1)| + |f(-1)-H_n(f;-1)|\} \\
&\quad \leqslant \omega(f;1-t) + 2(1-t)\,\omega\left(f;\sqrt{\frac{30}{11}\frac{1}{n^2}}\right) + |H_n(f;t)-H_n(f;1)| \\
&\quad \leqslant \omega(f;1-t^2) + 2(1-t^2)\,\omega\left(f;\sqrt{\frac{30}{11}\frac{1}{n^2}}\right) + |H_n(f;t)-H_n(f;1)| \\
&\quad \leqslant \omega(f;\sqrt{1-t^2}/n) + 4(1-t^2)\,\omega(f;1/n^2) + |H_n(f;t)-H_n(f;1)| \\
&\quad \leqslant \omega\left(f;\frac{\sqrt{1-t^2}}{n}\right) + 4(1-t^2)\left(1+\frac{1}{n\sqrt{1-t^2}}\right)\omega\left(f;\frac{\sqrt{1-t^2}}{n}\right) \\
&\qquad + |H_n(f;t)-H_n(f;1)| \\
&\quad \leqslant [1+8\sqrt{1-t^2}/n]\,\omega(f;\sqrt{1-t^2}/n) + |H_n(f;1)-H_n(f;t)| \\
&\quad \leqslant [1+8/n^2]\,\omega(f;\sqrt{1-t^2}/n) + |H_n(f;1)-H_n(f;t)| \\
&\quad \leqslant 1{,}08\omega(f;\sqrt{1-t^2}/n) + |H_n(f;1)-H_n(f;t)|, \qquad \text{if} \quad n \geqslant 10. \qquad (2.5)
\end{aligned}
$$

Now from (2.1) it follows that

$$
\begin{aligned}
&|H_n(f;t)-H_n(f;1)| \\
&\quad \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |f(\cos(\arccos t+v)) - f(\cos v)|\, K_{3n-3}(v)\,dv \\
&\quad \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} \omega(f;|\cos(\arccos t+v)-\cos v|)\, K_{3n-3}(v)\,dv \\
&\quad \leqslant \omega\left(f;\frac{\sqrt{1-t^2}}{n}\right)\left[1+\frac{n}{\pi\sqrt{1-t^2}}\int_{-\pi}^{\pi}|\cos(\arccos t+v)\right. \\
&\qquad \left. -\cos v|\,K_{3n-3}(v)\,dv\right] \\
&\quad \leqslant \omega\left(f;\frac{\sqrt{1-t^2}}{n}\right)\left[1+\frac{n}{\pi\sqrt{1-t^2}}\int_{-\pi}^{\pi}|(1-t)\cos v\right. \\
&\qquad \left. +\sqrt{1-t^2}\sin v|\,K_{3n-3}(v)\,dv\right]
\end{aligned}
$$

$$\leqslant \omega\left(f;\frac{\sqrt{1-t^2}}{n}\right)\left[1+\frac{n}{\pi\sqrt{1-t^2}}\int_{-\pi}^{\pi}(1-t)\cos vK_{3n-3}(v)\,dv\right.$$
$$\left.+\frac{n}{\pi}\int_{-\pi}^{\pi}|\sin v|\,K_{3n-3}(v)\,dv\right]$$
$$\leqslant \omega\left(f;\frac{\sqrt{1-t^2}}{n}\right)\left[1+\frac{(1-t)}{\pi(1-t^2)}\int_{-\pi}^{\pi}K_{3n-3}(v)\,dv\right.$$
$$\left.+\frac{2n}{\pi}\int_0^{\pi}vK_{3n-3}(v)\,dv\right]$$
$$\leqslant \omega\left(f;\frac{\sqrt{1-t^2}}{n}\right)\left[2+\frac{2n}{\pi}\int_0^{\pi}vK_{3n-3}(v)\,dv\right]$$

with

$$\frac{2n}{\pi}\int_0^{\pi}vK_{3n-3}(v)\,dv$$
$$=\frac{20}{\pi(11n^4+5n^2+4)}\int_0^{\pi}v\left(\frac{\sin nv/2}{\sin v/2}\right)^2\left(\frac{\sin nv/2}{\sin v/2}\right)^4dv$$
$$\leqslant\frac{20}{\pi(11n^4+5n^2+4)}\int_0^{\pi}\frac{nv^2\pi^2}{2v^2}\left(\frac{\sin nv/2}{\sin v/2}\right)^4dv$$
$$=\frac{20n\pi^2}{2\pi(11n^4+5n^2+4)}\int_0^{\pi}\left(\frac{\sin nv/2}{\sin v/2}\right)^4dv$$
$$=\frac{10n\pi^2}{2(11n^4+5n^2+4)}\,\frac{2}{\pi}\int_0^{\pi}\left(\frac{\sin nv/2}{\sin v/2}\right)^4dv$$
$$=\frac{10n\pi^2\,2n(2n^2+1)}{6(11n^4+5n^2+4)}\qquad\text{(cf. [2, p. 12])}$$
$$=\frac{20\pi^2(2n^4+n^2)}{6(11n^4+5n^2+4)}\leqslant\frac{20\pi^2\cdot 2.01n^4}{6\cdot 11n^4}\leqslant 6.1\qquad\text{if}\quad n\geqslant 10.$$

Hence

$$|H_n(f;t)-H_n(f;1)|\leqslant 8.1\omega(f;\sqrt{1-t^2}/n)\qquad\text{if}\quad n\geqslant 10.\tag{2.6}$$

Substituting this estimate into (2.5), we find that

$$|f(t)-\bar{H}_n(f;t)|\leqslant 10\omega(f;\sqrt{1-t^2}/n)\qquad\text{if}\quad n\geqslant 10.\tag{2.7}$$

Case (c) $\sqrt{1-t^2} \leqslant 1/n$, $t<0$. From (2.3) we obtain

$$
\begin{aligned}
f(t) - \bar{H}_n(f; t) &\\
= [f(t) - f(-1)] &+ [H_n(f; -1) - H_n(f; t)] \\
&+ ((1+t)/2)\{[f(-1) - H_n(f; -1)] + [H_n(f; 1) - f(1)]\}.
\end{aligned}
$$

On account of the fact that $1+t \leqslant 1-t^2$ for negative t we get again

$$|f(t) - \bar{H}_n(f; t)| \leqslant 10\omega(f; \sqrt{1-t^2}/n) \qquad \text{if} \quad n \geqslant 10$$

by means of a method analogous to the one used in Case (b).

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